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The Evolution of Probability Distributions: From Simple to Complex Parameter Spaces

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Abstract: This study evaluates the impact of number of parameters on the probability distribution. In this study, we have added one, two and three parameters in a single parameter exponential distribution and study the validation and precision of the probability models due to the added parameters. The primary objective of this research is to know about the impact of higher number of parameters on validation and precision of the probability models. We have generated a sample dataset of size hundred from two parameter exponential distribution newly formulated by adding a shape parameter to classical exponential distribution. For all models defined in this study, probability density curves and hazard rate curves are plotted and found that the models having more parameters are more flexible and valid compared to models having a smaller number of parameters. Plotting P-P and Q-Q plots is used to validate the models and estimate their parameters using the maximum likelihood function. Models with higher parameters have better validity, as seen by P-P and Q-Q plots. Calculations are made to determine the Akaike, Bayesian, Corrected Akaike, and Hannan-Quinn information criteria for model selection. Information criteria show that the model with larger number of parameters is more valid and flexible. To test the goodness of fit, Anderson darling test, Kolmogrov Smirnov test, and Cramer-von Mises methods are used. The R programming language is used to carry out all of the graphical and mathematical computations.

Keywords: Exponential distribution, Parameters Probability distribution, Model formulation, Validation, Maximum likelihood estimation

Introduction

Probability distributions are fundamental concepts in statistics and probability theory providing a mathematical framework for describing the likelihood of different

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outcomes in various scenarios. Probability distributions form the basis for statistical inference, modeling of random variables, and understanding the variability inherent in data. In other words, probability distributions provide a powerful framework for understanding and quantifying uncertainty in data. By characterizing the probabilities of possible outcomes, they facilitate statistical analysis, modeling, and inference across various disciplines, contributing to informed decision-making and robust scientific exploration. A probability model specifies the probabilities of the possible outcomes of a random variable. It describes how the total probability distributed among the possible values of the random variable. Many distributions are characterized by one or more parameters that determine their shape, location, and scale. For example, the parameters of Normal distribution are mean and standard deviations. Probability distributions are essential for making inferences about populations based on sample data, such as estimating parameters and testing hypotheses. They are used to model real-world phenomena and simulate outcomes in fields such as finance, engineering, and natural sciences. Probability models may have single more than one parameter.

Single parameter distributions are probability distributions that are characterized by a single parameter, which determines their shape, location, or scale. These distributions are foundational in statistics and probability theory, providing simple models for various types of random variables. Bernoulli distribution, Poisson distribution, Exponential Distribution, Geometric Distribution, and Uniform Distribution are some single parameter probability distributions.

Multi-parameter probability distributions are distributions that are characterized by more than one parameter. These distributions are used when a single parameter is insufficient to fully describe the variability or characteristics of a random variable. Normal Distribution, Beta Distribution, Dirichlet Distribution, gamma distribution, multinomial distributions are some commonly used multivariate probability distributions. Multi-parameter probability distributions are essential in statistical modeling and analysis, providing flexibility to describe complex data patterns and relationships. They are widely used across various disciplines, including engineering, economics, biology, and social sciences, for modeling real-world phenomena and making informed decisions under uncertainty. Understanding and correctly applying these distributions are crucial for conducting rigorous statistical inference and building reliable predictive models. Handling distributions with more parameters presents both theoretical and practical challenges in statistical modeling and analysis. While distributions with more parameters offer increased flexibility and potential for modeling complex data patterns, they also pose significant theoretical and practical challenges.

Addressing these challenges requires careful consideration of model complexity, data quality, computational resources, and effective communication of results to ensure robust and reliable statistical inference.

In literature, we can find numerous custom probability models having two or more than two parameters. These models are formulated using the classical probabilities models. The Classical probabilities models available in theory have more potential and are very useful in probability analysis. There are different types of classical probability models capturing the overall field and types of data available. In spite of various potentialities in classical probability models, there are some situations where these models could not analyze data precisely and the custom probability model formulated fits the data very well. Custom probability models have different shaped probability density function and hazard rate function that make it more flexible to fit for different new datasets.

Some of the newly generated probability distributions are, Exponentiated half logistic (Almarashi *et al.*, 2018), Exponentiated Chen distribution (Dey *et al.*, 2017), Lomax exponential distribution (Ijaz & Asim, 2019), Marshall-Olkin logistic-exponential distribution (Mansoor *et al.*, 2019), Generalized inverted generalized exponential distribution (Oguntunde & Adejumo, 2015), and Marshall-Olkin Kumaraswamy (Roshini and Thobias, 2017) etc. Some other modified classical probability distributions are logistic NHE distribution by (Chaudhary & Kumar,2020), Kumaraswamy alpha power inverted exponential distribution by (Thomas *et al.*, 2019) etc. Furthermore, Chaudhary *et al.* (2024) introduced Arctan exponential extension distribution, and new four-parameter extended exponential distribution created by (Hassan *et al.* ,2022). Chaudhary *et al.* (2024) also analyzed the air quality of Kathmandu using a probability model named the New Extended Kumaraswamy.

In literature, we can find many modifications of exponential distribution to generate new probability model. These modifications are done by adding some extra parameters, by merging two probability models, or by using family of probability distributions etc. Main objective behind the formulation of new models is to generate more flexible, valid and more reliable probability model that will cover wide range of data found in modern environment. Some of the generated probability distributions are; Generalized exponential proposed by (Gupta & Kundu, 2007), two-sided generalized exponential distribution was generated by (Korkmaz *et al.*,2015) etc. Similarly, Telee and Kumar (2023) introduced modified generalized exponential distribution. Other generalizations of exponential distribution are: beta exponential (Nadarajah & Kotz, 2006), Cauchy modified generalized exponential distribution (Chaudhary *et al.*, 2024) etc. Cordeiro and de Castro (2011) introduced Kumaraswamy exponential distribution. Merovci (2013) gave "Transmuted exponentiated exponential distribution", beta generalized exponential given by (Barreto Souza *et al.*, 2010), exponentiated exponential geometric by (Louzada *et al.*, 2014) are some models generated using exponential distribution. The gamma exponentiated exponential generated by (Ristic & Balakrishnan, 2011) as well as Kumaraswamy transmuted G-family of distribution was introduced by (Afify *et al.*,2016) are also generalization of exponential distribution. Generalized exponential distribution (Gupta & Kundu, 1999a), generalized Gompertz-Verhut family of distributions (Ahuja and Nash, 1967) and Lindley–exponential distribution (Bhati *et al.*, 2015) etc. are the modified distributions of exponential distributions.

Model Formulation

In this study, we have added one, two, or three shape as well as scale parameters to the classical exponential distribution to get modified exponential distribution having multi parameters probability distributions. A Poisson process, in which events happen continuously, independently, and at a constant average rate, describes the intervals between events. With a rate parameter $\lambda > 0$, the probability density function (PDF) of an exponential distribution is

$$f(x,\lambda) = \lambda e^{-\lambda x}; x > 0, \lambda > 0 \tag{1}$$

The exponential distribution's corresponding probability distribution function (or CDF) is

$$F(x,\lambda) = 1 - e^{-\lambda x}; x > 0, \lambda > 0 \tag{2}$$

One of the unique properties of the exponential distribution is that it is memory less. This means, time has already elapsed do not affect that the probability of an event occurring in the next interval of. Parameter λ controls the shape of the distribution. Higher λ means events occur more frequently, leading to shorter times between events. The reciprocal $1/\lambda$ gives the average time between events. The hazard rate of the exponential distribution is constant λ as given in eq. (3).

$$h(x) = \lambda = \text{a constant} \tag{3}$$

Similarly, the quantile function of the exponential distribution is given in eq. (4),

$$Q_X(u) = F_X^{-1}(u)$$

$$Q_X(u) = -\left(\frac{1}{\lambda}\right)\log(1-u); \quad 0 \le u \le 1 \tag{4}$$

The pdf plots in Figure 1(a) are displayed for various parameter values. Figure 1(b) also displays the hazard rate function of the exponential distribution. It is seen that there is less flexibility in pdf curves. The pdf curves are decreasing in shape while the hazard rate curve is constant.

During study, at first modification of Exponential distribution is done by adding a scale parameter β and is named as MEXP for simplicity. Equations (5) and (6), respectively, provide the pdf and cdf of the MEXP.

$$f(x,\beta,\lambda) = \lambda e^{-\lambda x e^{\beta x}} e^{\beta x}(1+\beta x); x>0, (\lambda,\beta)>0 \tag{5}$$

$$F(x,\beta,\lambda) = 1 - e^{-\lambda x e^{x\beta}}; x>0, (\lambda,\beta)>0 \tag{6}$$

Hazard rate function h(x) of MEXP is in eq. (7).

$$h(x) = \lambda e^{\beta x}(1+\beta x); x>0, (\lambda,\beta)>0 \tag{7}$$

Quantile function of MEXP is given by eq. (8)

$$Q_x(u) = \lambda x e^{\lambda x} + \log(1-u) = 0; \quad 0 \le u \le 1 \tag{8}$$

The pdf plots in figure 1(c) are displayed for various parameter values. Figure 1(d) also displays the hazard rate function of MEXP distribution. It is seen that there is a little more flexibility in pdf curve by adding a scale parameter. The pdf curve is decreasing as well as positively skewed in shape while the hazard rate curves are increasing and j shaped.

In next step, MEXP defined in eq. (5) is again modified by adding another shape parameter α. The new model formed has three parameters, α, β, and λ. The model so formed is named as Modified exponentiated exponential (MEEXP) model with pdf and cdf given by eq. (9) and (10) respectively.

$$f(x,\alpha,\beta,\lambda) = \lambda e^{\beta x} e^{-\lambda x^{\alpha} e^{x\beta}} \left(\alpha x^{\alpha-1} + x^{\alpha}\right); x>0, (\alpha,\lambda,\beta)>0 \tag{9}$$

$$F(x,\alpha,\beta,\lambda) = 1 - e^{-\lambda x^{\alpha} e^{x\beta}}; x>0, (\alpha,\lambda,\beta)>0 \tag{10}$$

The MEEXP's Hazard Rate Function h(x) is provided by Equation (11).

$$h(x) = \lambda e^{\beta x}\left(\alpha x^{\alpha-1} + x^{\alpha}\right); x>0, (\alpha,\lambda,\beta)>0 \tag{11}$$

Similarly, the quantile function of MEEXP is given by eq. (8)

$$Q_X(u) = \lambda x^{\alpha} e^{\lambda x} + \log\left(1-u\right) = 0; \quad 0 \le u \le 1 \tag{12}$$

The pdf plots in figure 1(e) are displayed for various parameter values. Hazard rate function of MEEXP is shown in figure 1(f). It is seen that there is more flexibility in pdf curves which is obtained as another parameter α is added to distribution MEXP. The pdf curve of MEEXP is decreasing as well as positively skewed, approximately normal as well as negatively skewed in shape while the hazard rate curves are bathtub, inverted j and j shaped.

MEEXP defined in eq. (9) is again modified by adding another shape parameter θ. The new model formed has four parameters, α, β, λ and θ. The model formed is named as *Theta power modified exponentiated exponential (TMEE) distribution* with pdf and cdf given by eq. (13) and (14) respectively.

$$f(x,\alpha,\beta,\lambda,\theta) = \lambda\theta e^{\beta x} e^{-\lambda x^{\alpha} e^{x\beta}} \left(1-e^{-\lambda x^{\alpha} e^{x\beta}}\right) \left(\alpha x^{\alpha} \quad + x^{\alpha}\right);\left(\alpha,\lambda,\beta,\theta\right) > 0, x > 0 \tag{13}$$

$$F(x,\alpha,\beta,\lambda,\theta) = \left(1-e^{-\lambda x^{\alpha} e^{x\beta}}\right)^{\theta}; x > 0, \left(\alpha,\beta,\lambda,\theta\right) > 0 \tag{14}$$

Hazard rate function h(x) of TMEE is given by eq. (11).

$$h(x) = f(x,\alpha,\beta,\lambda,\theta) = \lambda\theta e^{\beta x} e^{-\lambda x^{\alpha} e^{x\beta}} \left(1-e^{-\lambda x^{\alpha} e^{x\beta}}\right)^{\theta-1} \left[1-\left(1-e^{-\lambda x^{\alpha} e^{x\beta}}\right)^{\theta}\right]^{-1} \left(\alpha x^{\alpha-1} + x^{\alpha}\right); x > 0, \left(\alpha,\lambda,\beta,\theta\right) > 0 \tag{15}$$

Similarly, the quantile function of MEEXP is given by eq. (16).

$$Q_X(u) = \lambda x^{\alpha} e^{\lambda x} + \log\left(1-u^{1/\theta}\right) = 0; \quad 0 \le u \le 1 \tag{16}$$

The pdf graphs in figure 1(g) are displayed for various parameter values. Similarly, figure 1(h) displays the hazard rate function of the TMEE. It is seen that there is more flexibility in pdf curves due to addition of extra parameter θ than above three models. The pdf curve is decreasing as well as positively skewed, approximately normal and negatively skewed in shape while the hazard rate curves are bathtub, inverted j and j shaped.

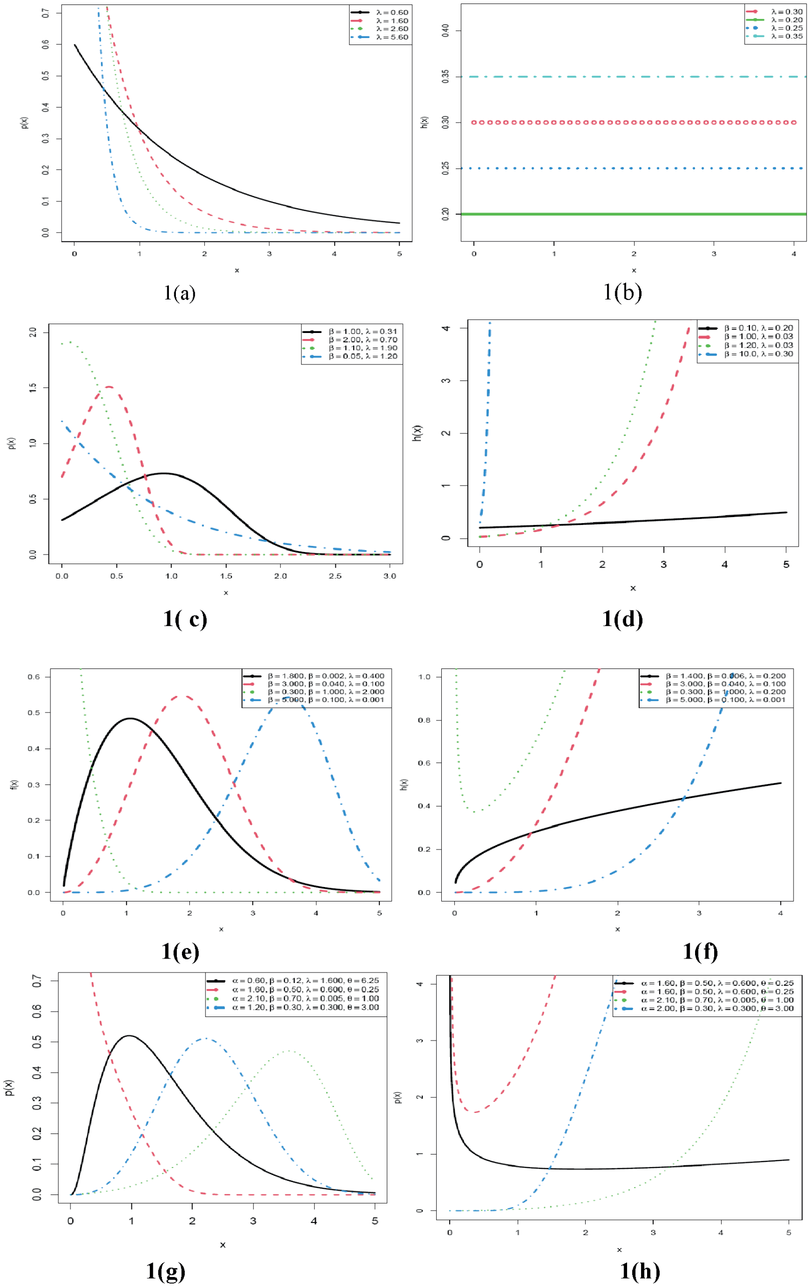

Figure 1: Pdf curves and hrf curves for all four models

Application to a Dataset

To test and compare the applicability of the models defined, we have generated a sample of 100 units from another modified exponential distribution. All the computational analysis are performed using R language (R Core Team, 2023). Here, we have modified the exponential distribution by taking the exponent of the classical exponential distribution defined in eq (1). The model defined has the following density and distribution functions.

$$f(x,\alpha,\lambda)=\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1};x>0,(\alpha,\lambda)>0;x>0,\lambda>0 \tag{17}$$

$$F(x;\alpha,\lambda)=\left(1-e^{-\lambda x}\right)^{\alpha};x>0,\ \alpha>0 \text{ and } \lambda>0 \tag{18}$$

Random deviate generation of the above model defined in eq (18) is given as

$$x=\left(-\frac{\log(1-p)}{\lambda}\right)^{1/\alpha};\quad 0\le \mathrm{p}\le 1 \tag{19}$$

Using random deviate generation (19), a dataset of sample size 100 is generated taking $\alpha = 1.5$ and $\lambda = 0.5$. The generated data are:

0.8149279, 0.7081576, 1.3721750, 0.2379471, 1.1692240, 1.2047875, 2.2375001, 0.9494139, 1.3575748, 0.5184395, 1.5670379, 2.6348508, 0.7565078, 1.0110477, 2.0221787, 1.6973791, 0.5940405, 0.9216821, 0.9258996, 1.7646891, 1.3306340, 1.8328188, 1.3369625, 1.9696944, 1.0590074, 0.5210229, 2.0531757, 2.6333746, 1.3639906, 0.7509051, 1.2154829, 3.0307538, 0.9026194, 3.3622235, 1.7809339, 2.6870363, 0.5409776, 1.5795679, 4.3667235, 0.4271675, 0.8638752, 2.5226751, 2.0830483, 2.3106765, 1.5066093, 1.2224080, 2.0946270, 2.6493287, 0.6007807, 0.8134766, 0.863595, 0.5811135, 0.6611388, 0.7448609, 1.4740500, 0.6990151, 0.4111236, 0.6487080, 1.4909380, 0.6087946, 1.1579994, 1.6310980, 3.4709733, 1.7203680, 1.1154669, 0.9222203, 1.1396491, 1.1160721, 0.6812751, 1.7779095, 1.0414815, 0.8575879, 1.4243491, 3.5971605, 1.6751528, 1.5661895, 2.4712999, 2.0714519, 2.3453791, 0.3327480, 1.1483694, 1.4958914, 2.9413479, 4.0427173, 0.1811041, 1.4384458, 1.9117822, 0.6890818, 0.7999356, 1.9123339, 2.8254542, 0.6053439, 0.9230077, 1.1226455, 2.8209644, 0.9911505, 1.2854314, 0.4152735, 0.1553307, 2.0592835.

Summary statistics of the generated data set are mentioned in table 1. Generated data is skewed as well as not normal.

Table 1: Summary statistics of generated data

Min.	*1st Qu*	*Median*	*Mean*	*3rd Qu*	*Max.*	*Skewness*	*Kurtosis*
0.1553	0.8101	1.2539	1.4594	1.9267	4.3667	1.028578	1.028578

Data Analysis

In this subsection, we have presented MLE method for the parameter estimation. Parameters of the entire proposed probability distributions are estimated. Estimated parameters and their respective standard errors of estimates (SE) are tabulated in table 2.

Table 2: Estimated Parameters and Standard Errors (SE)

Model	α	β	λ	θ
EXP	-	-	0.6852(0.0685)	-
MEXP	-	0.3560 (0.0701)	0.3176(0.0612)	-
MEEXP	1.7673(0.3120)	0.0021(0.1673)	0.4132(0.0957)	-
TMEE	0.6004(0.5358)	0.1184(0.1145)	1.6634(1.0966)	6.2427(9.2634)

Figure 2 shows the P-P and Q-Q plots for each of the four distributions. Figures 2(a) and 2(b) are the plots for exponential model. Plots shows that exponential distribution do not fit better to the given set of data. Figure 2(c) and 2(d) are the P-P and Q-Q plots for MEXP showing that the model fits a little bit better compared to exponential distribution for given data set. Similarly, figure 2(e) and 2(f) are the P-P and Q-Q plots for MEEXP model. The plots indicate that model fits given data set better compared to exponential model as well as MEXP. In figure 2(g) and 2(h), P-P and Q-Q plots for TMEE model are displayed. Plots shows that given data set fit the model far better compared to other three models.

To test the suitability and validity of the models, we have compared the log-likelihoods, BIC, CAIC, AIC and HQIC for each model and are mentioned in table 3. It is found that information criteria values decrease as the number of parameters increases for this particular dataset. This indicates that as number of parameters increases, the accuracy of the model increases.

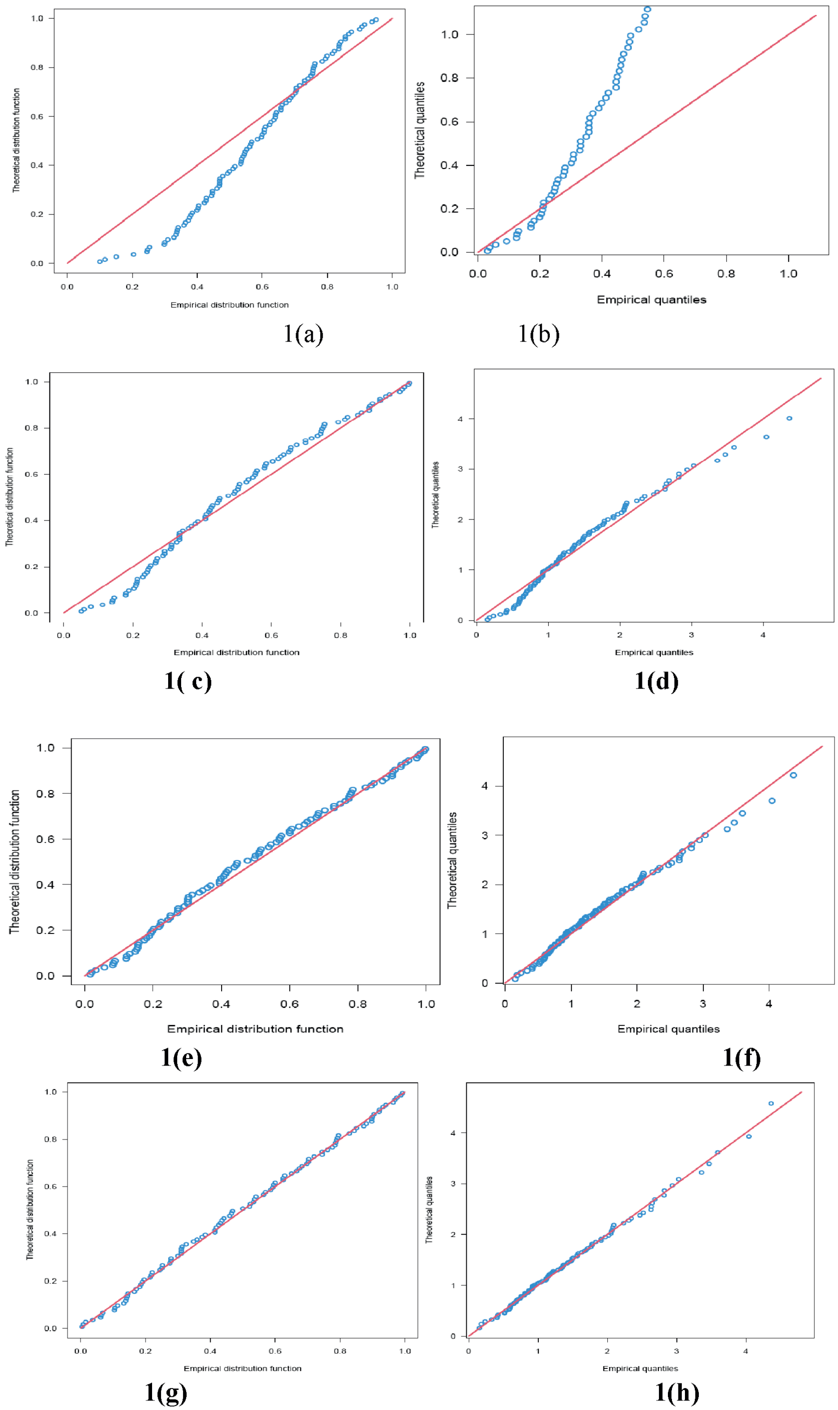

Figure 2: P-P plots and Q-Q plots for all four models

Table 3: Information criteria values for models

Model	LL	AIC	BIC	CAIC	HQIC
EXP	-137.8040	277.608	280.213	277.649	278.662
MEXP	-122.8215	249.643	254.853	249.767	251.752
MEEXP	-116.5312	239.062	246.878	239.312	242.225
TMEE	-115.0740	238.148	248.569	238.569	242.365

Kolmogorov- Smirnov (KS), Cramer's-von Mises (CVM) and Anderson Darling (AD) test statistics are compared. Test statistics and respective p values are in table 5. Goodness of fit test shows that as number of parameters increases from one to four, the test statistics values decrease with increment of p- values. This indicates that as number of parameters increases, model fits better for data set considered here.

Table 5: KS, CVM, and AD statistics and respective p values

Model	*KS (p-value)*	*CVM (p-value)*	*AD (p-value)*
EXP	0.228975(0.00006)	1.41322(0.00025)	7.79894(0.00015)
MEXP	0.109665(0.18036)	0.21352(0.24291)	1.57242(0.16015)
MEEXP	0.053859(0.93380)	0.05990(0.81572)	0.39700(0.85139)
TMEE	0.037329(0.99904)	0.01562(0.99952)	0.12476(0.99972)

Goodness of fits are also displayed graphically in figure 3. The histogram of given dataset and the fitted pdfs for all the four models are plotted in left panel of figure 3. It is clear from graph that the model with higher number of parameters fits the data well compared to the distribution with lesser number of parameters. Similarly, in right panel of the figure 3, we have plotted the empirical cdf against the fitted cdfs for all the models.

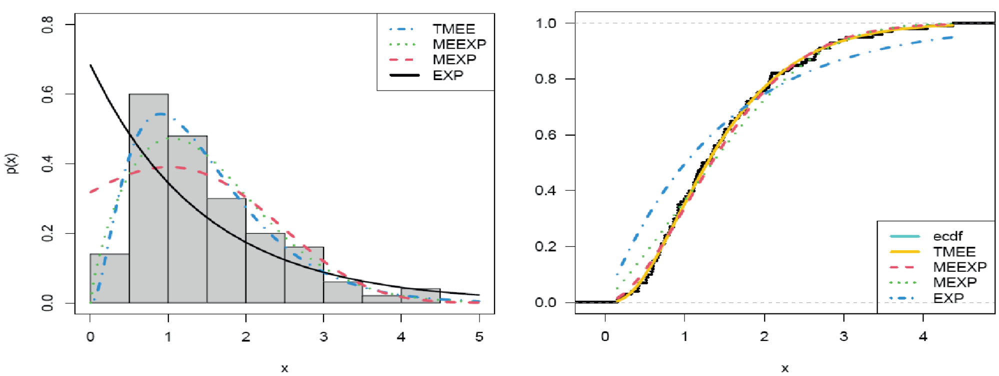

Figure 3: Histogram versus fitted density curves (Left) and Empirical cdf versus fitted cdfs

Conclusion

This article is based on the analysis of affect of number of parameters on the model validation and goodness of fit. We have considered here a single parameter exponential distribution and formed another three probability models adding one, two, or three parameters. A dataset is generated from another newly modified probability model and the information criteria values are calculated for all the models. Information criteria values show that as the number of parameters increases, its values decrease showing that addition of some extra scale and shape parameters make distribution more flexible and valid. Using maximum likelihood estimation, model parameters are estimated. Pdf plots and hazard rate curves also show that there is more flexibility in nature of curves as the number of parameters increases from one to four. The P-P and Q-Q plots of the models also verify that distribution with higher number of parameters fits data set well. Goodness of fit are checked using KS, CVM and AD test. Respective p-values as well as the test statistics show that as number of parameters increases from one to four, precision and validation of the model increases. All the graphics and the computational studies are performed using R programming.

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